# Group-Walk Random Graphs

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**Abstract**

We introduce a construction that gives rise to a variety of 'geometric' finite random graphs, and describe connections to the Poisson boundary, Naim's kernel, and Sznitman's random interlacements.

## 1   Introduction

The purpose of this paper is to introduce a new construction of 'geometric' finite random graphs, called Group-Walk Random Graphs (GWRGs from now on) and describe the rich connections to other objects, including the Poisson boundary, Naim's kernel, and Sznitman's random interlacements. GWRGs do not only yield new interesting examples of random graphs, but as we will argue, they can be thought of as a tool for studying groups.
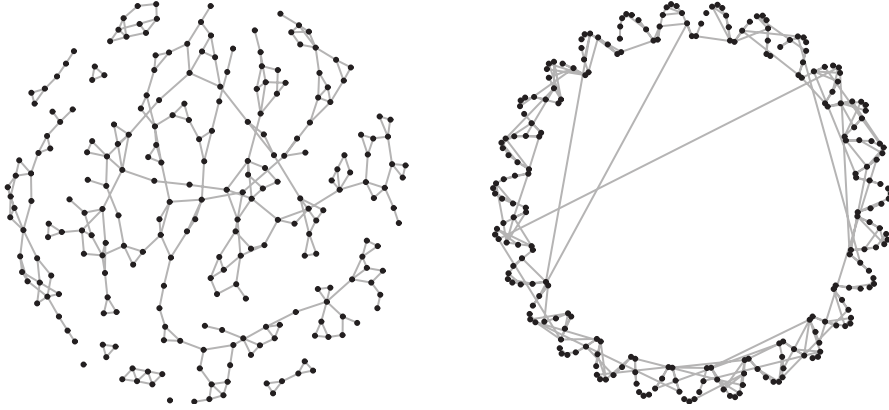


Figure 1: A sample GWRG produced by computer simulation by A. Janse van Rensburg. The host graph is a ternary tree, and $n = 5$.

We start by introducing the simplest special case of GWRGs. Let $G$ be an infinite homogeneous tree, rooted at a vertex $o$, called the *host* graph; we will later allow $G$ to be an arbitrary locally finite Cayley graph, or even a more general graph. Let $G_n := G[\{v \in V(G) \mid d(v, o) \le n\}]$ be the ball of radius $n$

centered at $o$, and define the boundary $\partial G_n$ to be the set $\{v \in V(G) \mid d(v,o) = n\}$ of vertices at distance exactly $n$ from $o$.

We construct a random graph $R_n$ as follows. The vertex set of $R_n$ is the deterministic set $\partial G_n$. The edge set of $R_n$ is constructed according to the following process. We start an independent simple random walk in $G_n$ from each vertex $v \in \partial G_n$, and stop it upon its first return to $\partial G_n$, letting $v^\dagger$ denote the vertex in $\partial G_n$ where this random walk was stopped. We then put an edge in $R_n$ joining $v$ to $v^\dagger$ for each $v \in \partial G_n$.

We could stop the construction here and declare $R_n$ to be our GWRG, but it is more interesting to consider the following evolution: let $R_n^1 := R_n$, and for $i = 2, 3, \ldots$ let $R_n^i$ be the union of $R_n^{i-1}$ with an independent sample of $R_n^1$; or in other words, $R_n^i$ is the random graph obtained as above when we start $i$ independent particles at each vertex in $\partial G_n$.

An important observation from [13] is that these random graphs $R_n^i$ have the following scale-invariance property. Let $C, D$ be two *branches* of our tree $G$, where a branch is a component of $G \backslash e$ for some edge $e$. Then, for any fixed $i$, we have

**Observation 1.1** ([13]). *The expected number of edges of $R_n^i$ from branch $C$ to branch $D$ converges as $n \to \infty$. The limit is always $> 0$, and it is finite if and only if $C \cap D = \emptyset$.*

This might at first sight look surprising, as the number of vertices of $R_n^i$ inside each of $C, D$ grows exponentially with $n$, yet no rescaling is involved in Observation 1.1.
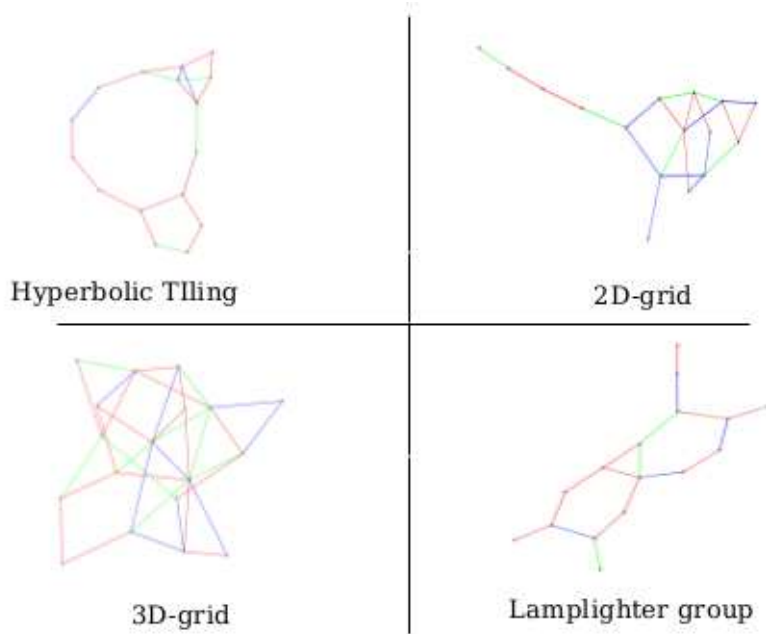


Figure 2: Computer simulations of GWRGs with 3 repetitions (coded by different colours) for different host graphs by C. Midgley.

The same construction can be repeated when instead of the binary tree the

host graph is any infinite graph. Then Observation 1.1 has a generalisation, but in order to formulate it we need the Poisson boundary $\mathcal{P}$: instead of the 'branches' of Observation 1.1 we have to talk about subgraphs 'converging' to a measurable subset of $\mathcal{P}$. This is explained in greater detail in Section 2, where we also elaborate more on the general construction of GWRGs and their variants.

The construction of GWRGs was motivated by a measure space introduced in [13], called the effective conductance measure, which is closely related to the Poisson boundary. It is a generalisation of effective conductance for electrical networks, and it is important for the study of Dirichlet harmonic functions on infinite graphs. More details about this measure are given in Section 3.

I expect that GWRGs can unify many existing models of geometric random graphs, while introducing new ones, and offer new tools for analysing them including the Poisson boundary, as well as the notion of graphons [5]. Conversely, GWRGs provide an additional tool for indirectly studying groups, just like random walks; see Section 8.3 for more.

The study of random graphs is currently one of the most active branches of graph theory. By far the most studied random graph model is that of Erdős & Renyi (ER) [8], in which every pair of vertices is joined with an edge with the same probability, and independently of each other pair.

In recent years, many models of *geometric* random graphs have been emerging [14, 18]. The idea now is to embed the set of vertices (possibly randomly) into a geometric space —usually the euclidean or hyperbolic plane, and their higher dimensional analogs— and then to independently join each pair of vertices with a probability that decays as the distance between the vertices in the underlying space grows.

One advantage of these geometric random graphs compared to the ER model is that they can approximate real-life networks much more realistically, but they are also of great theoretical interest given the impact of the ER model. A disadvantage is that there is an infinity of such models, obtained by varying the underlying geometry, the way the points are embedded, and the connection probability as a function of distance, and no canonical choice is available.

I like thinking of GWRGs as geometric random graphs, where the underlying geometry is a Cayley graph $G$ of an arbitrary finitely generated group. Although there is a huge variety for the underlying 'geometry', the construction is in a sense canonical, and many tools for their analysis are available and are discussed here.

This paper is written in survey style although the material reviewed is quite new and partly under development, the main aim being to make the open problems of the project accessible to other researchers willing to get involved. A lot of the material is drawn from the paper [13] which is still in progress. New here is the definition of GWRGs and some observations about them.

## 2    The general construction of GWRGs

In the Introduction we chose the host graph $G$ to be a tree, the reason being that Observation 1.1 is easier to state in that case. Let us now consider the general case where the host graph $G$ is arbitrary, and see how Observation 1.1

generalises, which will lead us to the definition of the effective conductance measure.

The construction of $R_n^i$ can be repeated verbatim, except that the number of particles we start at a vertex $v \in \partial G_n$ in round $i$ is equal to the vertex degree $d_{G_n}(v)$ of $v$ in the ball $G_n$; the reason will become apparent later. However, we could instead of starting exactly $d_{G_n}(v)$ particles at $v$ in round $i$, start a random number of particles following some distribution with expectation $d_{G_n}(v)$, the most natural candidate being the Poisson distribution; the following discussion remains valid for this variant of GWRGs.

Observation 1.1, which is easy to prove in the case of trees, now becomes a substantial theorem, but in order to formulate it we need to involve the concept of the Poisson boundary of $G$ to extend the above notion of branch in the correct way.

The Poisson boundary of an infinite (transient) graph $G$ is a measurable, Lebesgue-Rohlin, space $\mathcal{P} = \mathcal{P}(G)$, endowed with a family of probability measures $\{\mu_v \mid v \in V(G)\}$, such that every bounded harmonic function $h : V(G) \to \mathbb{R}$ can be represented by integration on $\mathcal{P}$: we have $h(v) = \int_{\mathcal{P}} \hat{h} d\mu_v(\eta)$ for a suitable boundary function $\hat{h} : \mathcal{P} \to \mathbb{R}$. This can be thought of as a discrete version of Poisson's integral representation formula $h(z) = \int_0^1 \frac{1-|z|^2}{|e^{2\pi i \theta} - z|^2} \hat{h}(\theta) d\theta = \int_0^1 \hat{h}(\theta) d\nu_z(\theta)$, recovering every continuous harmonic function $h : \mathbb{D} \to \mathbb{R}$ in terms of its boundary values $\hat{h} : \mathbb{S}^1 \to \mathbb{R}$, except that we replaced $\mathbb{D}$ with a transient graph.

Triggered by the work of Furstenberg [10] who introduced the concept, the study of the Poisson boundary of Cayley graphs has grown into a very active research field; see [9] for a survey including many references. Although there is a straightforward abstract construction of the Poisson boundary $\mathcal{P}(G)$ of any Cayley graph, given a concrete $G$ it is desirable to identify $\mathcal{P}(G)$ with a geometric boundary. This pursuit can however be very hard, although some general criteria are available.

As an example, we remark that the Poisson boundary of a regular tree can be identified with its set of ends, and the Poisson boundary of a regular tessellation of the hyperbolic plane can be identified with its circle at infinity. More generally, the Poisson boundary of any non-amenable, bounded-degree, Gromov-hyperbolic graph coincides with its hyperbolic boundary [1, 2]. The Poisson boundary of an 1-ended bounded-degree planar graph can be identified with a circle [11].

Now back to Observation 1.1, letting $G$ be an arbitrary transient graph, we consider measurable subsets $X, Y$ of $\mathcal{P}(G)$. One can associate with these sets sequences of vertex sets $(X_n)_{n \in \mathbb{N}}, (Y_n)_{n \in \mathbb{N}}$, where $X_n, Y_n \subseteq \partial G_n$ in the above notation, such that for random walk on $G$ from any starting vertex, the events of converging to $X$ and visiting infinitely many of the $X_n$ coincide up to a set of measure zero, and similarly for $Y$ and the $Y_n$. The first statement of Observation 1.1 generalises:

**Theorem 2.1** ([13]). *The expected number of edges of $R_n^i$ from $X_n$ to $Y_n$ converges as $n \to \infty$.*

For the second statement of Observation 1.1 we remark that the limit is infinite when $X \cap Y$ has positive measure, and there are, rather rare, interesting cases where the limit is infinite independently of the choice of $X, Y$ as long as

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they have positive measure: any lamplighter graph over a transient graph has this property [13].

In order to understand why Theorem 2.1 (or Observation 1.1) is true, it is helpful to consider the well-known relationship between random walks and electrical networks as introduce by Doyle & Snell [7]. Think of $G_n$ as an electrical network with boundary nodes $\partial G_n$, at which we impose a constant potential $v(b) = 1$ for every $b \in \partial G_n$. This Dirichlet problem has the unique trivial solution $v(x) = 1$ for every $x \in V(G_n)$. Now, the aforementioned relationship tells us that the solution to any such Dirichlet problem can be obtained as follows: we start $d_{G_n}(b)v(b)$ random walk particles at each boundary node $b$, and stop them upon their first re-visit to $\partial G_n$. Then letting $v(x)$ be the expected number of visits to $x$ by all those particles divided by the degree $d(x)$ solves the Dirichlet problem [12]. But as our Dirichlet problem has constant boundary values, we then expect $d(x)$ visits to each interior vertex $x$. This implies that if we start $d_{G_n}(b)$ random walkers at each vertex $b \in \partial G_n$, stop them upon their first re-visit to $\partial G_n$, and observe the parts of their trajectories inside $G_m$ for $m < n$, then the situation we observe will be similar as if we had performed the same process on $G_m$ instead of $G_n$. This is the central observation for the proof of Theorem 2.1.

Theorem 2.1 will be crucial in the next section.

# 3    The effective conductance measure $\mathcal{C}$

Theorem 2.1 relates our GWRGs to the Poisson boundary, but in fact the connection is more intricate. Before elaborating on this, we recall Douglas' classical formula

$$E(h) = \int_0^{2\pi} \int_0^{2\pi} (\hat{h}(\eta) - \hat{h}(\zeta))^2 \Theta(\eta, \zeta) d\eta d\zeta,$$

expressing the (Dirichlet) energy of a harmonic function $h$ on the unit disc $\mathbb{D}$ in the complex plain from its boundary values $\hat{h}$ on the circle $\mathbb{S}^1 = \partial\mathbb{D}$. The physical intuition here is that $E(h)$ is the power dissipated by a circular metal plate, on the boundary circle of which some potential is imposed by an external source or field.

A discrete variant of this formula is the following, expressing the energy dissipated by a finite electrical network with a set $B$ of boundary nodes, in terms of the voltages at $B$ and certain 'effective conductances' relative to $B$ (see [13] for details).

$$E(h) = \sum_{a,b \in B} (h(a) - h(b))^2 C_{ab}. \tag{1}$$

In [13] we prove the following statement, providing a general formula for the Dirichlet energy of harmonic functions on a graph, which is similar to Douglas' formula

**Theorem 3.1** ([13]). *For every transient graph $G$, there is a measure $\mathcal{C}$ on $\mathcal{P}(G)^2$ such that for every harmonic function $h : V(G) \to \mathbb{R}$ with boundary function $\hat{h}$, the energy $E(h)$ equals*

$D(\hat{h}) := \int_{\mathcal{P}}^2 \left( \hat{h}(\eta) - \hat{h}(\zeta) \right)^2 \mathcal{C}(\eta, \zeta).$

This measure $\mathcal{C}$, which we call the *effective conductance measure*, can be thought of as a continuous analogue of effective conductance in a finite electrical network due to the similarity of the above formula with (1); in the next section we will elaborate more on this.

The construction of $\mathcal{C}$ is based on Theorem 2.1: we set the value $\mathcal{C}(X, Y)$ to be the limit value returned by that theorem, and apply Caratheodory's extension theorem to show that this defines a measure on $\mathcal{P}(G)^2$. This explains the relationship between GWRGs and $\mathcal{C}$, and also my motivation in introducing the former.

## 4   Doob's formula and Naim's kernel

Theorem 3.1 was motivated by a similar result of Doob [6], which generalises Douglas' formula to arbitrary Green spaces. I will refrain from repeating the definition of Green spaces here, which can be found in [6], and suffice it to say that they generalise Riemannian manifolds. In a sense, our Theorem 3.1 is a discrete version of Doob's result. But rather than working with the Poisson boundary (which first appeared the year after Doob's paper [10]) Doob worked with the Martin boundary, which is closely related to the Poisson boundary (the latter can be defined as the support of the former, but the former is also endowed with a topology). Moreover, Doob's approach to the effective conductance measure was different to ours: rather than working with the measure, Doob was working with its density, namely the Naim kernel. Let me explain this further, as it will be of interest later.

Naim [17] introduced the formula

$$\Theta(x, y) := \frac{G(x, y)}{G(x, o)G(o, x)}$$

where $G(\cdot, \cdot)$ denotes Green's function, and $x, y$ are points of a Green space $X$. The same formula however makes perfect sense when $x, y$ are points of a transient graph $G$, in which case $G(x, y)$ denotes the expected number of visits to $y$ by random walk from $x$.

Naim proved that $\Theta(x, y)$ can be extended to pairs of points $\{\eta, \zeta\}$ in the Martin boundary of $X$, by taking a limit

$$\Theta(\eta, \zeta) := lim_{x \to \eta, y \to \zeta}\Theta(x, y). \tag{2}$$

The convergence of this limit is only clear when one of $x, y$ is fixed and the other converges to a point $\eta$ in the boundary, as it reduces to the well-known convergence of the Martin kernel $K(x, y) := \frac{G(x,y)}{G(o,y)}$ [22]. The two-sided convergence is a puzzling fact, and in fact is not true everywhere but 'almost everywhere': the exact statement proved by Naim [17] is too technical to state here precisely. It involves Cartan's fine topology. As far as I know, this convergence has not been proved for graphs; we will return to this issue below.

Our intuitive interpretation of $\Theta(\eta, \zeta)$ is that it denotes the 'effective conductance density' between the boundary points $\eta, \zeta$. To support this intuition, we remark in [13] that if $x, y$ are boundary vertices of a finite electrical network, then $\Theta(x, y)$ does indeed coincide with the effective conductance between $x$ and

$y$ if the Green function $G$ is defined with respect to random walk killed at the boundary.

Doob's formula for the Dirichlet energy of a harmonic function $h$ on $X$, with boundary extension $\hat{h}$ on $M(X)$ reads

$$D(\hat{h}) := \int_{M(X)^2} \left(\hat{h}(\eta) - \hat{h}(\zeta)\right)^2 \Theta(\eta,\zeta) d\mu_o(\eta) d\mu_o(\zeta).$$

Notice the similarity to the formula of Theorem 3.1 and Douglas's (1).

As we show in [13], our effective conductance measure $\mathcal{C}$ is absolutely continuous with respect to the square of harmonic measure $\mu_o$ on $\mathcal{P}$, and so we can define a kernel $\Theta'$ on $\mathcal{P}^2$ by the Radon-Nykodym derivative $\frac{\partial \mathcal{C}}{\partial \mu_o^2}(\eta,\zeta)$. The above discussion suggests that $\Theta'$ should coincide with $\Theta$ if $\Theta$ can be defined by a limit similar to (2). Now instead of trying to imitate Naim's proof of the convergence of (2), which is rather technical, we propose the following

**Problem 4.1.** *Let $G$ be a transient graph and $o \in V(G)$. Let $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ be independent simple random walks from $o$. Then $\lim_{n,m \to \infty} \Theta(x_n, y_m)$ exists almost surely.*

This is seemingly easier than trying to prove the convergence of (2), since for example it is easier to prove the Martingale convergence theorems than Fatou's theorem. If it is true, then one could further ask

**Problem 4.2.** *Let $G$ be a transient graph and $o \in V(G)$. Let $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ be independent simple random walks from $o$, and $X, Y$ measurable subsets of $\mathcal{P}(G)$. Then $\mathcal{C}(X,Y) = \mathbb{E} \lim_{n,m \to \infty} \mu_{x_n}(X) \mu_{y_n}(Y) \Theta(x_n, y_m)$, where $\mathcal{C}$ denotes the effective conductance measure.*

The factors $\mu_{x_n}(X), \mu_{y_n}(Y)$ in the above limit essentially 'condition' the random walks to converge to $X, Y$ respectively. This last problem is motivated by our expectation that $\Theta' = \Theta$.

## 5   Random interlacements

Recall that we defined $\mathcal{C}(X,Y)$ as the limit of the expected number of edges of our GWRG $R_n^1$ from $X_n$ to $Y_n$. In doing so, we were only interested in the starting and finishing vertices of our random walks, ignoring the exact trajectories inside the host graph $G$. In this section we remark that the distributions of these trajectories converge, in a certain sense, to the intensity measure of the Random Interlacement model as introduced by Sznitman [19] for $G = \mathbb{Z}^d$ and generalised to arbitrary transient $G$ by Teixera [20].

Given a transient graph $G$, the Random Interlacement on $G$ is a Poisson point process on the space of 2-way infinite trajectories in $G$ modulo the time shift. It is governed by a $\sigma$-finite measure $\nu$ on $(W^*, \mathcal{W}^*)$, where $W^*$ denotes the set of equivalence classes of 2-way infinite walks in $G$ with respect to the time shift, and $\mathcal{W}^*$ is the canonical sigma-algebra on $W^*$. We remark that this measure $\nu$ can also be obtained from the process we used to construct our GWRG $R_n^1$ as follows (the original definition of Sznitman will be given below).

Let $C$ be a cylinder set of $\mathcal{W}^*$ defined by a finite walk $Z$, i.e. is the set of 2-way infinite walks containing $Z$ as a subsequence. Let $\mathcal{P}_n$ be the random

process from the definition of $R_n^1$, that is, a collection of particles performing random walk starting at $\partial G_n$ and stopped upon the first revisit to $\partial G_n$, with $d_{\partial G_n}(x)$ particles started at each $v \in \partial G_n$. We set

$$\mu(C) := \lim_n \mathbb{E}\{\# \text{ of trajectories in } \mathcal{P}_n \text{ containing } Z \text{ as a subwalk }\}, \qquad (3)$$

and extend $\mu$ to a measure on $(W^*, \mathcal{W}^*)$ using e.g. Caratheodory's extension theorem. It turns out [13] that

$$\mu = \nu$$

An interesting consequence of this is the following relation between the effective conductance measure $\mathcal{C}$ and the Random Interlacement intensity measure, $\nu$

**Corollary 5.1.** *For every two measurable subsets $X, Y$ of $\mathcal{P}(G)$, we have*

$$\mathcal{C}(X, Y) = \nu(W_{XY}^*),$$

*where $W_{XY}^*$ denotes the set of elements of $W^*$ all initial subwalks of which meet infinitely many $X_n$ (as defined in Section 2) and all final subwalks of which meet infinitely many $Y_n$.*

In order to explain why this is true we need to recall the definition of the intensity measure $\nu$ from [20].

To begin with, given a finite subset $K$ of $V(G)$, we define the *equilibrium measure $e_K$* on $K$ by

$$e_K(x) = \mathbb{1}_{x \in K} c_x \mathbb{P}_x[A \mid \text{ random walk does never return to } K \,],$$

where $c_x$ is the vertex degree of $x$.

Let $\pi^*$ be the canonical projection from the set of 2-way infinite walks $W$ to $W^*$. Then $\nu$ is defined as the unique measure on $(W^*, \mathcal{W}^*)$ satisfying, for every finite subset $K$ of $V(G)$,

$$\mathbb{1}_{W_K^*} \cdot \nu = \pi^* \circ Q_K. \qquad (4)$$

(Another way to state this formula is $\nu(\mathbb{1}_{W_K^*} \cdot A) = Q_K(\pi^{*-1}(A))$, where $\pi^{*-1}(A)$ returns those walks in the $\pi^*$-preimage of $A$ that enter $K$ at time 0 for the first time.)

Here, $Q_K$ is a finite measure on the space $W_K$ of 2-way infinite walks meeting $K$, given by the formula

$$Q_K[(X_{-n})_{n \geq 0} \in A, X_0 = x, (X_n)_{n \geq 0} \in B] = \mathbb{P}_x[A \mid \text{no return to } K] e_K(x) \mathbb{P}_x[B],$$

where $A, B$ are measurable subsets of the space $W_+$ of 1-way infinite walks. Let

me explain why $\nu$ coincides with $\mu$ as claimed above. The main idea is to think of the equilibrium measure $e_K(x)$, which is proportional to the probability of escaping $K$, as the probability for a random walker coming from 'infinity' to enter $K$ at $x$; this intuition is justified by the reversibility of our walks, and coming from infinity can be made precise using the process used to construct GWRGs (and $\mu$).

To make this more precise, let $G_n$ denote the ball of radius $n$ around $o$, and suppose that $K \subset V(G_n)$ for some $K \subset V(G)$. Let $G_n^*$ be the graph obtained

from $G$ by contracting the complement of $G_n$ into a single vertex $*_n$. Then the reversibility of our random walk implies that, for every $x \in K$, letting $\mathbb{P}_y$ denote the law of a random walk $X_0, X_1, \ldots X_\tau$ from $y$ stopped the first time $\tau > 0$ when $X_\tau \in K \cup \{*_n\}$, we have

$$c_x \mathbb{P}_x[X_\tau = *_n] = c_{*_n} \mathbb{P}_{*_n}[X_\tau = x].$$

Now note that, by the transience of $G$, the limit as $n$ goes to infinity of the left hand side converges to $e_K(x)$, while the limit of the right hand side is closely related to our definition of $\mu$.

Random interlacements have been studied extensively, and have found interesting applications in the study of the vacant set for random walk on discrete tori []. This connection to the Poisson boundary has apparently not been observed before.

## 6   Graphons

Graphons were recently introduced [5] as a notion of limit for sequences of dense finite graphs, and they have already had a seminal impact on combinatorics. Formally, a graphon is a symmetric, measurable function $w : [0,1]^2 \to [0,1]$.

Every graphon naturally gives rise to a family of random graphs $G_n$ on $n$ vertices for every $n \in \mathbb{N}$: we sample $n$ independent, uniformly distributed points from $[0,1]$ to be the vertices of $G_n$, and join vertices $x, y$ with probability $w(x,y)$.

Graphs sampled this way are dense, i.e. have average degree of order $n$. But a variant of the above sampling method introduced in [4] produces sparse graphs.

Now note that formally, our effective conductance measure $\mathcal{C}$ is similar to a graphon, as it is a measure on a square. In particular, we can sample a random graph from it. We expect these graphs to be closely related to our GWRGs.

**Problem 6.1.** *Show that for every (transient) Cayley graph $G$, the sequence of corresponding GWRGs converges to a sparse graphon in the sense of [4].*

## 7   Simulation data

Computer simulations on GWRG for certain concrete host graphs $G$, performed by Chris Midgley [16], suggest that certain properties are heavily influenced by $G$, while convergence with $n$ is fast enough that simulation data can help make explicit predictions.

In most simulations the host graph was the 2 or 3-dimensional grid $\mathbb{Z}^2$ or $\mathbb{Z}^3$, the infinite binary tree $T_2$, a hyperbolic planar graph $T^\circ$ obtained from the binary tree by adding a cycle $C_n$ joining the vertices at distance $n$ from the root for every $n$, and the lamplighter graph $L$ over $\mathbb{Z}$.

The outcomes of these simulations, with 10,000 random graphs $R_n^1$ generated in each case for $n$ up to 8 (where the exponential growth of $T^\circ$ and $L$ become computationally demanding), suggest the following.

- The number of isolated vertices of $R_n^1$ is asymptotically proportional to the number of vertices $|\partial G_n|$ of $R_n^1$. The same is true for the number of

components of $R_n^1$. The corresponding leading coefficients are very similar (possibly converging to the same number) when $G$ is $\mathbb{Z}^2$ or $T^\circ$, slightly different when $G = L$, and very different when $G = \mathbb{Z}^3$.

- The expected diameter of the largest connected component of $R_n^1$ seems to be proportional to $\log(|\partial G_n|)$.

In a further experiment of [16] on the infinite binary tree $T_2$, random graphs $R_n^i$ were generated for $2 \leq n \leq 8$ for all values of $i$ until the first time that $R_n^i$ becomes connected. The data suggest (quite clearly) that the average time $i$ till connectedness is roughly $0.26n$, i.e. linear in $n$. (The exact outcome was $i = (0.26 \pm 0.003)k + (0.90 \pm 0.02)$ with an adjusted $R^2$ value of 0.9993.)

# 8 Further Problems

## 8.1 Dirichlet harmonic functions

Let $G$ be a graph on which all harmonic functions with finite Dirichlet energy are constant; the class of such graphs is denoted by $\mathcal{O}_{HD}$. By Theorem 3.1, we can introduce the following trichotomy for such graphs:

(i) $\mathcal{P}(G)$ is trivial (in other words, $G$ has the Liouville property), or

(ii) $\mathcal{P}(G)$ is not trivial, and $\mathcal{C}(X, Y) = \infty$ for every two measurable $X, Y \subseteq \mathcal{P}(G)$, or

(iii) none of the above holds, but the integral of Theorem 3.1 is infinite for every boundary function $\hat{h}$.

Recall that the property of being in $\mathcal{O}_{HD}$ is quasi-isometry invariant for graphs of bounded degree [21], and therefore, given a group $\Gamma$, it is independent of the choice of the Cayley graph of $\Gamma$. A well-known open problem asks whether the Liouville property too is independent of the choice of the Cayley graph of $\Gamma$. The above trichotomy suggests the following refinement of that problem for groups in $\mathcal{O}_{HD}$

**Problem 8.1.** *Let $G$ be a Cayley graph in $\mathcal{O}_{HD}$. Do all finitely generated Cayley graphs of the group of $G$ have the same type with respect to the above trichotomy?*

We remark that Cayley graphs of all three types exist: $\mathbb{Z}^3$ is of type (i), lamplighter graphs over any transient Cayley graph are of type (ii) [13], while all tessellations of hyperbolic 3-space $\mathbb{H}^3$ are of type (iii).

## 8.2 GWRGs

None of the results suggested by the simulations of Section 7 have been proved rigorously, and it would be interesting to do so, especially if the methods involved can be applied to large families of host graphs. Results on the expected number of components or isolated vertices of $R_n^1$ could be within reach.

The interesting meta-problem is to find natural properties of the GWRGs that are universal in the sense that they are true independently of the host graph

$G$ (under some restriction, e.g. $G$ being vertex-transitive), as well as properties that do depend on $G$ and the dependence can be explained. For example, what can be said about $R_n$ if the host $G$ is hyperbolic?

Erdős Renyi random graphs $G(n,p)$ are known to display a sharp threshold for their connectedness at the value $p = \ln n/n$ for the presence of each edge. This threshold coincides with the threshold for having an isolated vertex [8]. This motivates the question of whether similar behaviour is observed for GWRGs. To make this more precise, define the random variable $\tau_n := \min\{i \mid G_n^i$ is connected$\}$. Our first question is how concentrated $\tau_n$ is (we interpret high concentration as a sharp phase transition). The second question is whether the threshold for connectedness coincides with that for absence of isolated vertices: letting $\tau_n^* := \min\{i \mid G_n^i$ has no isolated vertices$\}$,

**Problem 8.2.** *Is $\lim_n \mathbb{E}\tau_n = \lim_n \mathbb{E}\tau_n^*$?*

This might depend on the host graph $G$, and the answer might be positive for every 'nice' host, e.g. any Cayley graph.

For any host graph $G$, and $n \in \mathbb{N}$, consider the ball $G_n$ as a random rooted graph $G_n^*$ by rooting it at a uniformly chosen vertex in the boundary $\partial G_n$. Then, as suggested by Gourab Ray (private communication), if $G_n^*$ converges in the local weak sense, then our GWRG $R_n^i$ should converge for every $i$ in the Benjamini-Schramm sense [3]. The relationship between this limit and the host graph could be interesting. (But even if $G_n^*$ does not converge, it will have sub-sequential limits and the situation is not less interesting.)

A particular case of interest is where $G$ is the grid $\mathbb{Z}^d$, and we start particles at $\partial G_n$ according to the Poisson distribution in the construction of $R_n^i$. The aforementioned limit coincides then with the long-range percolation model as defined e.g. in [15] (this connection was again noticed by Gourab Ray).

It would be interesting to compare our GWRGs induced by certain simple Cayley graphs, like tessellations of the hyperbolic plane, to other geometric random graphs from the literature:

**Problem 8.3.** *Show that the standard geometric random graph constructions can be obtained as special cases of GWRGs by choosing an appropriate underlying graph.*

## 8.3 Groups

Our main objective is to understand the interplay between typical properties of GWRGs and their host groups. In particular, we have

**Problem 8.4.** *Which properties of the random graphs are determined by the group of the host graph and do not depend on the choice of a generating set?*

A further problem in this vein is

**Problem 8.5.** *Are the properties of transience, Liouvilleness, existence of harmonic Dirichlet functions, and amenability on the host Cayley graph detectable by the asymptotic behaviour of the corresponding GWRGs?*

Answers to these problems would make GWRG a tool for studying group-theoretical questions, on a par with the study of random walks on groups.

# 9 Conclusions

We introduced a construction of 'geometric' random graphs (GWRGs), and established strong links to the Poisson boundary of their host graphs, the effective conductance measure and Naim's kernel, and to random interlacements. This project will be successful if we can exploit these connection in order to make conclusions about one of these objects by studying the other. Another major aim is to relate GWRGs to existing models of geometric random graphs. Last but not least, we would like to understand the effect of various properties of the host group on the typical graph-theoretic properties of GWRGs.

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